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INCIDENCE ALGEBRAS AND COALGEBRAS OF DECOMPOSITION STRUCTURES

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In this paper a few relationships between a Decomposition Structure and its Incidence Coalgebra and Algebra are studied. In particular, some results about the so-called Isomorphism Problem for Incidence Algebras of Moebius Categories are generalized. Moreover, we determine conditions under which all endomorphisms and derivations of an Incidence Algebra are continuous with respect to the finite topology.

Introduction

The notion of Incidence Algebra was introduced by G.-C. Rota and others in order to supply a unified algebraic setting for a wide class of problems of enumerative combinatorics. In particular this has made possible a more simple and general formulation of the classical Möbius Inversion. The structure of Incidence Algebra was originally associated with locally finite posets and monoids with the finite factorization property. Afterwards, this procedure was extended to the set of morphisms of a decomposition–finite category. This includes not only the posets and monoids, but also other objects closely connected to enumerative combinactorics. For the purpose of clarifying the connection between the set $\mathbb{S}$ of morphisms of one category and its Incidence Algebra $A(\mathbb{S})$, the study of the so-called Incidence Coalgebra $C(\mathbb{S})$, of which $A(\mathbb{S})$ is the dual algebra, has proved useful. This same notion of coalgebra has furthermore indicated the possibility of utilising this technique in the study of a wider class of objects than that of the categories. One may, in fact, resort to the notion of Incidence Coalgebra whenever one finds a "decomposition structure" $\mathbb{S}$; that is to say, whenever one finds a set S together with a rule, for cutting each of its elements in two, which satisfies certain "reasonable" conditions according to the definition given by Joyal in [7]. In the present work we mean to study the reciprocal relations between $\mathbb{S}$, $C(\mathbb{S})$ and $A(\mathbb{S})$. We observe that in this context the Incidence Coalgebra $C(\mathbb{S})$, in virtue of its relationship of duality with the Incidence Algebra $A(\mathbb{S})$, has an equivalent function to that carried out by the so-called finite topology in $A(\mathbb{S})$.

A classical problem is as follows: to what extent does our knowledge of $A(\mathbb{S})$ or $C(\mathbb{S})$ enable us to reconstruct the decomposition structure $\mathbb{S}$? A few partial solutions to this problem, obtained by Leroux [8] in the case of the categories, are here extended to a class of decomposition structures which we have called regular. In the same way, the Propositions 3.18 and 3.34 are generalizations of analogous results relating to categories and posets (see [8] and [1]). In actual fact such propositions state that all automorphisms and derivations of an Incidence Algebra of a finitely generated decomposition structure are continuous with respect to finite topology. Moreover, we shall see how a simple condition regarding the cardinality of a particular subset S_0 of S allows us to extend the first of the former propositions to all the endomorphisms of $A(\mathbb{S})$.

1. Decomposition law on a set

Let S be a set. Let $\mathbb{N}[S\times S]$ denote the free abelian monoid generated by $S\times S$. A *decomposition law* on S is a pair of applications

$$d: S \longrightarrow \mathbb{N}[S\times S]$$

$$s \mapsto d(s) = \sum_{q,r} \begin{bmatrix} s \\ q, r \end{bmatrix} (q, r)$$

$$e: S \longrightarrow \mathbb{N}$$

where the coefficients $\left[{s \atop q,r}\right]$, usually called *section coefficients*, satisfy the following equations:

$$\sum_r \begin{bmatrix} s \\ q, r \end{bmatrix}\begin{bmatrix} r \\ t, v \end{bmatrix} = \sum_r \begin{bmatrix} r \\ q, t \end{bmatrix}\begin{bmatrix} s \\ r, v \end{bmatrix} \tag{1.1}$$

$$\sum_q \begin{bmatrix} s \\ q, r \end{bmatrix} e(q) = \sum_q \begin{bmatrix} s \\ r, q \end{bmatrix} e(q) = \delta_r^s \tag{1.2}$$

The section coefficient $\left[{s \atop q,r}\right]$ counts the different ways in which d cuts the element $s\in S$ into the same ordered pair $(q, r)\in S\times S$. In the following, the triple $\mathbb{S}=(S, d, e)$ will be referred to as a *decomposition structure*.

Let us denote $\left[{s \atop q,t,v}\right]$ the common value of both sides of 1.1. More generally, let us put:

$$\begin{bmatrix} s \\ r \end{bmatrix} := \delta_r^s$$

and, for every $n>1$,

$$\begin{bmatrix} s \\ r_1 \cdots r_{n+1} \end{bmatrix} := \sum_t \begin{bmatrix} s \\ r_1 \cdots r_{n-1}, t \end{bmatrix}\begin{bmatrix} t \\ r_n, r_{n+1} \end{bmatrix}$$

Reiterating 1.1, for any sequence of integers $1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n$, we have:

$$\begin{bmatrix} s \\ r_1 \cdots r_{n+1} \end{bmatrix} = \sum_{q_1 \cdots q_{k+1}} \begin{bmatrix} q_1 \\ r_1 \cdots r_{i_1} \end{bmatrix} \cdots \begin{bmatrix} q_{k+1} \\ r_{i_k+1} \cdots r_{n+1} \end{bmatrix} \begin{bmatrix} s \\ q_1 \cdots q_{k+1} \end{bmatrix}$$

If $[{}^{\;s}_{r_1 \cdots r_n}] > 0$, the n-tuple $(r_1 \cdots r_n)$ is called a *decomposition of degree n* of s; $[{}^{\;s}_{r_1 \cdots r_n}]$ is the number of ways we can cut s into the ordered n-tuple $(r_1, \ldots, r_n)$. The decomposition $(r_1, \ldots, r_n)$ is called a *proper decomposition* if $d(r_i) \neq (r_i, r_i)$ for $1 \leqslant i \leqslant n$. The supremum, in $\mathbb{N} \cup \{\infty\}$, of the set of degrees of proper decompositions of an element $s \in S$ is called the *length of s* and denoted $l(s)$. It is easy to check that $l(s) = 0$ if and only if $d(s) = (s, s)$. For every $n \in \mathbb{N}$ we put:

$$S_{(n)} := \{s \in S \mid l(s) = n\} \quad \text{and} \quad S_n := \bigcup_{k \leqslant n} S_{(n)}.$$

Owing to the following proposition S_n, together with the restriction of the decomposition law, may be considered as a "substructure" $\mathbb{S}_n$ of $\mathbb{S}$.

Proposition 1.3. *If* $[{}^{\;s}_{q,r}] > 0$, *then* $l(q) + l(r) \leqslant l(s)$.

Proof. Observe that if $(q_1, \ldots, q_h)$ and $(r_1, \ldots, r_k)$ are proper decompositions of q and r, respectively, then

$$\begin{aligned} \begin{bmatrix} s \\ q_1 \cdots q_h, r_1 \cdots r_k \end{bmatrix} &= \sum_{u,v} \begin{bmatrix} s \\ u, v \end{bmatrix} \begin{bmatrix} u \\ q_1 \cdots q_h \end{bmatrix} \begin{bmatrix} v \\ r_1 \cdots r_k \end{bmatrix} \\ &\geqslant \begin{bmatrix} s \\ q, r \end{bmatrix} \begin{bmatrix} q \\ q_1 \cdots q_h \end{bmatrix} \begin{bmatrix} r \\ r_1 \cdots r_k \end{bmatrix} > 0. \end{aligned}$$

As a consequence $(q_1 \cdots q_h, r_1 \cdots r_k)$ is a proper decomposition of s. □

In the study of the decomposition structure $\mathbb{S}$, a central role is played by the *neutral elements* (i.e. the elements $s \in S$ such that $e(s) = 1$), as shown by the following propositions, due essentially to Joyal [7].

Proposition 1.4. *For each* $s \in S$, *there exists a unique pair of neutral elements* $\partial_0(s)$ *and* $\partial_1(s)$ *such that both* $[{}^{\;\;s}_{\partial_0(s), s}]$ *and* $[{}^{\;\;s}_{s, \partial_1(s)}]$ *are positive.*

We have:

$$\begin{bmatrix} s \\ \partial_0(s), s \end{bmatrix} = \begin{bmatrix} s \\ s, \partial_1(s) \end{bmatrix} = 1.$$

Proposition 1.5. *If* $s \in S_0$, *then s is a neutral element of* $\mathbb{S}$.

Proposition 1.6. *The following statements are equivalent*:
(i) s *is neutral*;
(ii) $e(s) > 0$;
(iii) $\partial_0(s) = s$ (*resp*. $\partial_1(s) = s$).

Proposition 1.7. *If* $[{}^{\,s}_{q,r}] > 0$ *and* $e(r) = 1$ (*resp*. $e(q) = 1$), *then* $s = q$ *and* $r = \partial_1(s)$ (*resp*. $s = r$ *and* $q = \partial_0(s)$).

Proposition 1.8. *If* $[{}^{\,s}_{q,r}] > 0$, *then* $\partial_0(s) = \partial_0(q)$, $\partial_1(q) = \partial_0(r)$, $\partial_1(r) = \partial_1(s)$.

If q, r are neutral elements and U is a subset of S then we put:

$$U(q, r) = \{s \in U \mid \partial_0(s) = q \quad \text{and} \quad \partial_1(s) = r\}.$$

The *length n graph* of a decomposition structure $\mathbb{S}$ is the direct graph whose vertices are the neutral elements of $\mathbb{S}$ and whose edge-set is $S_{(n)}$. The arrow $s \in S(n)$ is directed from $\partial_0(s)$ to $\partial_1(s)$. Assuming the whole S as edge-set, we get a new directed graph, the so-called *associated graph* of $\mathbb{S}$. Obviously, the associated graphs of two decomposition structures $\mathbb{S}$ and $\mathbb{T}$ are isomorphic if and only if there exists a bijection $\theta : S \to T$ such that $\theta(\partial_0(s)) = \partial_0(\theta(s))$ and $\theta(\partial_1(s)) = \partial_1(\theta(s))$, for every $s \in S$. Moreover if $l(\theta(s)) = l(s)$ for every $s \in S$, then we say that $\mathbb{S}$ and $\mathbb{T}$ have *isomorphic presentations*. In this case $\mathbb{S}$ and $\mathbb{T}$ have isomorphic length n graphs for every $n \in \mathbb{N}$.

A decomposition structure $\mathbb{S}$ is said to be *hereditarily finite* if each $s \in S$ admits a finite number of proper decompositions.

Proposition 1.9. *The following statements are equivalent*:
(i) *every element of* $\mathbb{S}$ *has a finite length*;
(ii) $\mathbb{S}$ *is a hereditarily finite decomposition structure*;
(iii) *every neutral element has length zero*; *moreover if, for some* s, $[{}^{\,s}_{r,s}] > 0$ *then* r *is a neutral element.*

Proof. (i)$\Rightarrow$(ii). It is sufficient to show that, for each $s \in S$, the number of decompositions of any degree is finite. Obviously, this is true for degree 2. Arguing by induction, from

$$\begin{bmatrix} s \\ r_1 \cdots r_{n+1} \end{bmatrix} = \sum_t \begin{bmatrix} t \\ r_1 \cdots r_n \end{bmatrix} \begin{bmatrix} s \\ t,\ r_{n+1} \end{bmatrix}$$

we obtain that the number of sequences $(r_1, \ldots, r_{n+1})$ such that $[{}^{\quad s}_{r_1 \cdots r_{n+1}}] > 0$ is finite.

(ii)$\Rightarrow$(iii). (See also [7], Theorem 7, p. 67). Let s be a neutral element; if

$d(s) \neq (s, s)$, then from

$$\begin{bmatrix} s \\ s \cdots s \end{bmatrix} \geqslant \begin{bmatrix} s \\ s, s \end{bmatrix} \cdots \begin{bmatrix} s \\ s, s \end{bmatrix} > 0$$

we deduce that $(s, \ldots, s)$ is a proper decomposition of s. Thus s admits proper decompositions of any length. This contradicts (ii). Suppose, now, $[{}^{s}_{r,s}] > 0$. If s is a neutral element then $d(s) = (s, s)$; so $r = s$. On the other hand, if neither s nor r are neutral elements then

$$\begin{bmatrix} s \\ r \cdots r, s \end{bmatrix} \geqslant \begin{bmatrix} s \\ r, s \end{bmatrix} \cdots \begin{bmatrix} s \\ r, s \end{bmatrix} > 0.$$

It follows that s admits proper decompositions of any length which again contradicts (ii).

(iii)$\Rightarrow$(i). Let m denote the number of decompositions of degree 2 of an element $s \in S$ and let $(r_1, \ldots, r_n)$ be a proper decomposition of s. Since

$$\begin{bmatrix} s \\ r_1 \cdots r_n \end{bmatrix} = \sum_t \begin{bmatrix} t \\ r_1, r_2 \end{bmatrix} \begin{bmatrix} s \\ t, r_3 \cdots r_n \end{bmatrix} > 0,$$

there exists $q_2 \in S$ such that

$$\begin{bmatrix} q_2 \\ r_1, r_2 \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} s \\ q_2, r_3 \cdots r_n \end{bmatrix} > 0.$$

Repeating the same argument one obtains a sequence $q_1, \ldots, q_{n-1}$ of elements of S such that $q_1 = r_1$, $[{}^{q_{i+1}}_{q_i, r_{i+1}}] > 0$, $1 \leqslant i \leqslant n-2$, and $[{}^{s}_{q_i, r_{i+1} \cdots r_n}] > 0$, $1 \leqslant i \leqslant n-1$. Similarly one may obtain a sequence $p_1, \ldots, p_{n-1}$ such that $[{}^{s}_{q_i, p_i}] > 0$ and $[{}^{p_i}_{r_{i+1} \cdots r_n}] > 0$, $1 \leqslant i \leqslant n-1$. Since $i > j$ implies $[{}^{q_i}_{q_j, r_{j+1} \cdots r_i}] > 0$, all this proves that there exists an element t_{ji} such that $[{}^{q_i}_{q_j, t_{ji}}] > 0$ as well as $[{}^{t_{ji}}_{r_{j+1} \cdots r_i}] > 0$. Since $d(r_i) \neq (r_i, r_i)$, from (iii) we deduce $d(t_{ji}) \neq (t_{ji}, t_{ji})$ and $q_i \neq q_j$. Therefore the pairs (q_i, p_j) get $n-1$ distinct decompositions of s. Thus $n \leqslant m + 1$. $\square$

A decomposition structure is associated with any small category C such that the set $\{(q, r) \mid q, r \in \mathrm{Mor}(C), q \circ r = s\}$ is finite for every $s \in \mathrm{Mor}(C)$. The decomposition law is now:

$$\begin{aligned} d : S &\longrightarrow \mathbb{N}[S \times S] \\ s &\mapsto \sum_{r \circ q = s} (q, r) \end{aligned}$$

$$\begin{aligned} e : S &\longrightarrow \mathbb{N} \\ s &\mapsto \begin{cases} 1 & \text{if } s \text{ is an identity of } C \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

When such a decomposition structure is hereditarily finite, C is said to be a *Möbius Category*. Decomposition structures associated with small categories have been studied in [3, 8].

We shall see that many results obtained for Möbius Categories are also true for a larger class of decomposition structures, the class of regular decomposition structures.

Let $\mathbb{S}=(S, d, e)$ be a decomposition structure and $\mathbb{Z}[A]$ denote the free $\mathbb{Z}$-module generated by the set A. Let us put

$$\begin{aligned}\bar{\delta}^n_S : \mathbb{Z}[S] &\longrightarrow \mathbb{Z}[S\times\cdots\times S]\\ s &\mapsto \sum \begin{bmatrix} s \\ r_1 \cdots r_{n+1}\end{bmatrix}(r_1, \ldots, r_{n+1})\end{aligned}$$

where the sum ranges over all proper decompositions $(r_1, \ldots, r_{n+1})$ of degree $n+1$ of s.

$\mathbb{S}$ is said to be an *n-regular decomposition structure* if $\operatorname{Ker}(\bar{\delta}^n_S)=\mathbb{Z}[S_n]$. Moreover, $\mathbb{S}$ is said to be a *regular decomposition structure* if $\mathbb{S}$ is an n-regular decomposition structure for each $n\in\mathbb{N}$.

All the decomposition structures associated with small categories are regular.

$\mathbb{S}$ is said to be a *finitely generated decomposition structure* if $\mathbb{S}$ is a regular hereditarily finite decomposition structure and, for each pair $q, r\in S_0$,

(i) $S_{(1)}(q, r)$ is finite;

(ii) $[q, r]=\{s\in S_0 \mid \text{there exist } u, v, w\in S \text{ with } [{}^{\,u}_{v,w}]>0 \text{ and } \partial_0(u)=q,\ \partial_0(w)=s$ and $\partial_1(u)=r\}$ is finite.

Proposition 1.10. *If $\mathbb{S}$ is a finitely generated decomposition structure then, for each pair $q, r\in S_0$, the set $S_{(n)}(q, r)$ is finite.*

Proof. The proposition is trivial for $n=1$. Let us prove it for $n+1$ under the assumption that it is true for n. To this aim, consider the set U of all proper decompositions of degree $n+1$ of the elements of $S_{(n+1)}(q, r)$. Notice that, because of the regularity of $\mathbb{S}$, the cardinality of $S_{(n+1)}(q, r)$ is less than that of U. Let $(r_1, \ldots, r_{n+1})$ be a proper decomposition of $s\in S_{(n+1)}(q, r)$. Since

$$\begin{bmatrix} s \\ r_1 \cdots r_{n+1}\end{bmatrix}=\sum_t \begin{bmatrix} s \\ t,\ r_{n+1}\end{bmatrix}\begin{bmatrix} t \\ r_1 \cdots r_n\end{bmatrix}>0,$$

there exist $u\in[q, r]$ and $v\in S_{(n)}(q, u)$ such that $r_{n+1}\in S_{(1)}(u, r)$ and $[{}^{\;v}_{r_1\cdots r_n}]>0$. By inductive hypothesis, we deduce that for every $u\in[q, r]$ the set of the proper decompositions of all the elements of $S_{(n)}(q, u)$ is finite. It follows that U, and then $S_{(n+1)}(q, r)$, is finite. $\square$

2. Incidence coalgebras

Let K be a field of characteristic zero. Each decomposition structure $\mathbb{S} = (S, d, e)$ allows us to define a coalgebra over K. Let us associate a variable x_s to each $s \in S$ and denote $K[S]$ the K-vector space spanned by x_s's. Owing to 1.1 and 1.2 the linear maps:

$$\begin{aligned} \Delta_S : K[S] &\longrightarrow K[S] \otimes K[S] \\ x_s &\mapsto \sum_{q,r} \begin{bmatrix} s \\ q, r \end{bmatrix} x_q \otimes x_r \end{aligned} \tag{2.1}$$

$$\begin{aligned} \varepsilon_S : K[S] &\longrightarrow K \\ x_s &\mapsto e(s) \end{aligned} \tag{2.2}$$

can be considered as a diagonalization and counít map respectively in a coalgebra structure (see [10]). This coalgebra, denoted $C(\mathbb{S}) = (K[S], \Delta_S, \varepsilon_S)$, is said to be the *Incidence Coalgebra* of $\mathbb{S}$.

By Proposition 1.3, we have $\Delta_S(K[S_n]) \subseteq K[S_n] \otimes K[S_n]$ for each $n \in \mathbb{N}$. Thus if we restrict both Δ_S and ε_S to $K[S_n]$, we obtain a subcoalgebra $C(\mathbb{S}_n)$ of $C(\mathbb{S})$. Another class of subcoalgebras of $C(\mathbb{S})$ is obtained considering, for each pair $p, q \in S_0$, subsets of S of the kind $S_{(n+1)}(p, q) \cup S_n$.

Consider the maps:

$$\begin{aligned} \Delta_S^n : K[S] &\longrightarrow \bigotimes^{n+1} K[S] \\ x_s &\mapsto \sum_{r_1 \cdots r_{n+1}} \begin{bmatrix} s \\ r_1 \cdots r_{n+1} \end{bmatrix} x_{r_1} \otimes \cdots \otimes x_{r_{n+1}} \end{aligned}$$

and

$$\begin{aligned} \bar{\Delta}_S^n : K[S] &\longrightarrow \bigotimes^{n+1} K[S] \\ x_s &\mapsto \sum \begin{bmatrix} s \\ r_1 \cdots r_{n+1} \end{bmatrix} x_{r_1} \otimes \cdots \otimes x_{r_{n+1}} \end{aligned}$$

where the last sum ranges over all the proper decompositions of degree $n+1$ of S. Obviously, $\Delta_S = \Delta_S^1$ and $\bar{\delta}_S^n$ is the restriction of $\bar{\Delta}_S^n$ to $\mathbb{Z}[S]$. We also have:

$$\bar{\Delta}_S^n = \left(\bigotimes^{n+1} P_S \right) \circ \Delta_S^n$$

where P_S denotes the linear map

$$\begin{aligned} P_S : K[S] &\longrightarrow K[S] \\ x_s &\mapsto \begin{cases} 0 & \text{if } s \in S_0 \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

It is plain that if $\mathbb{S}$ is an n-regular decomposition structure then $\mathrm{Ker}(\bar{\Delta}_S^n) = K[S_n]$.

An element $c \in C(\mathbb{S})$ is said to be a *grouplike element* if $\Delta_S(c) = c \otimes c$ and $\varepsilon_S(c) = 1$. The following proposition tells us that neutral elements of a hereditarily finite decomposition structure $\mathbb{S}$ are precisely the grouplike elements of the associated coalgebra $C(\mathbb{S})$.

Proposition 2.3. *Let $\mathbb{S}$ be a hereditarily finite decomposition structure. An element $c \in C(\mathbb{S})$ is grouplike if and only if $c = x_s$ for $s \in S_0$.*

Proof. If $s \in S_0$ then x_s is a grouplike element of $C(\mathbb{S})$. Conversely, let $c = \sum_s k^s x_s$ be a grouplike element of $C(\mathbb{S})$. We have:

$$\sum_{q,r} \left(\sum_s k^s \begin{bmatrix} s \\ q, r \end{bmatrix} \right) x_q \otimes x_r = \sum_s k^s \Delta_S(x_s) = \Delta_S(c) = c \otimes c = \sum_{q,r} k^q k^r x_q \otimes x_r.$$

Thus $\sum_s k^s [{}^{\,s}_{q,r}] = k^q k^r$ for each pair $q, r \in S$. Since $\mathbb{S}$ is a hereditarily finite decomposition structure, $\max\{l(s) \mid x_s \text{ occurs in } c\}$ is a non-negative integer n. So, if x_t occurs in c and $l(t) = n$ then $\sum_s k^s [{}^{\,s}_{t,t}] = k^t k^t \neq 0$. Hence, there exists $u \in S$ such that $k^u \neq 0$ and $[{}^{\,u}_{t,t}] \neq 0$. It follows by Proposition 1.3, $l(u) \geqslant 2n$, i.e. $n = 0$; so if x_v occurs in c then $v \in S_0$. Since grouplike elements of a coalgebra over a field are linearly independent (see [10]), we have $c = k^s x_s$, $s \in S_0$. Owing to $\varepsilon_S(c) = 1$, we deduce $k^s = 1$. This completes the proof. □

Let $C(\mathbb{S})$, $C(\mathbb{T})$ be Incidence Coalgebras. We recall that a linear map $\phi : C(\mathbb{S}) \to C(\mathbb{T})$ is a *coalgebra map* if $\Delta_T \circ \phi = (\phi \otimes \phi) \circ \Delta_S$ and $\varepsilon_T \circ \phi = \varepsilon_S$.

From this definition, by induction on n, we deduce:

$$\Delta_T^n \circ \phi = \left(\bigotimes^{n+1} \phi \right) \circ \Delta_S^n.$$

A similar formula holds for $\bar{\Delta}^n$ only if $\mathbb{T}$ is a hereditarily finite decomposition structure. We shall prove this after the following proposition.

Proposition 2.4. *Let $\mathbb{T}$ be a hereditarily finite decomposition structure. If $\phi : C(\mathbb{S}) \to C(\mathbb{T})$ is a coalgebra map then, for every $s \in S_0$, there exists $t \in T_0$ such that $\phi(x_s) = x_t$.*

Proof. Let $s \in S_0$; then $\Delta_T(\phi(x_s)) = (\phi \otimes \phi) \circ \Delta_S(x_s) = \phi(x_s) \otimes \phi(x_s)$ and $\varepsilon_T(\phi(x_s)) = \varepsilon_S(x_s) = 1$. Thus $\phi(x_s)$ is a grouplike element of $C(\mathbb{T})$. Hence, by Proposition 2.3, $\phi(x_s) = x_t$ where $t \in T_0$. □

As a consequence of Proposition 2.4, if $\mathbb{T}$ is a hereditarily finite decomposition structure, every coalgebra map $\phi : C(\mathbb{S}) \to C(\mathbb{T})$ gives rise to a map $\phi_0 : S_0 \to T_0$

where $\phi_0(s)=t \Leftrightarrow \phi(x_s)=x_t$. In the following we shall always denote ϕ_0 the associated map with the coalgebra map ϕ.

Corollary 2.5. *Let* $\mathbb{T}$ *be a hereditarily finite decomposition structure. If* $\phi : C(\mathbb{S}) \to C(\mathbb{T})$ *is a coalgebra map then*

$$\bar{\Delta}^n_T \circ \phi = \left(\bigotimes^{n+1} P_T\right) \circ \left(\bigotimes^{n+1} \phi\right) \circ \bar{\Delta}^n_S.$$

Proof.

$$\bar{\Delta}^n_T \circ \phi = \left(\bigotimes^{n+1} P_T\right) \circ \Delta^n_T \circ \phi = \left(\bigotimes^{n+1} P_T\right) \circ \left(\bigotimes^{n+1} \phi\right) \circ \Delta^n_S$$

$$= \left(\bigotimes^{n+1} (P_T \circ \phi)\right) \circ (\bar{\Delta}^n_S + \Delta^n_S - \bar{\Delta}^n_S).$$

Now, by Proposition 2.4 and by the definition of P_T, $(\bigotimes^{n+1}(P_T \circ \phi)) \circ (\Delta^n_S - \bar{\Delta}^n_S)$ is the zero map. Thus $\bar{\Delta}^n_T \circ \phi = (\bigotimes^{n+1} P_T) \circ (\bigotimes^{n+1} \phi) \circ \bar{\Delta}^n_S$. $\square$

Proposition 2.6. *Let* $\mathbb{T}$ *be a hereditarily finite decomposition structure and* $\phi : C(\mathbb{S}) \to C(\mathbb{T})$ *a coalgebra map. If* $l(s)$ *is finite and if* x_t, $t \in T$, *occurs in* $\phi(x_s)$ *then there exists a decomposition* (p, q, r) *of* s *such that* x_t *occurs in* $\phi(x_q)$ *and* $\partial_0(t) = \phi_0(\partial_0(q))$, $\partial_1(t) = \phi_0(\partial_1(q))$.

Proof. Notice that if we put $\phi(x_s) = \sum_{t \in T} \phi^t_s x_t$ for each $s \in S$, from

$$(\phi \otimes \phi \otimes \phi)(\Delta^2_S(x_s)) = \Delta^2_T(\phi(x_s))$$

we obtain:

$$\sum_{a,b,c} \phi^\alpha_a \phi^\beta_b \phi^\gamma_c \begin{bmatrix} & s & \\ a, & b, & c \end{bmatrix} = \sum_p \phi^p_s \begin{bmatrix} & p & \\ \alpha, & \beta, & \gamma \end{bmatrix}$$

for each $s \in S$ and each triple $\alpha, \beta, \gamma \in T$. So, assuming $\alpha = \partial_0(t)$, $\beta = t$, and $\gamma = \partial_1(t)$, we have:

$$\sum_{a,b,c} \phi^{\partial_0(t)}_a \phi^t_b \phi^{\partial_1(t)}_c \begin{bmatrix} & s & \\ a, & b, & c \end{bmatrix} = \phi^t_s.$$

Now by Proposition 2.4, the proposition holds when $l(s)=0$. Arguing by induction on $l(s)=n$, let us assume that it holds for the elements of S with a length less than n. Given the former equality, if $\phi^t_s \neq 0$ there exists a decomposition (u, v, w) of s such that $\phi^{\partial_0(t)}_u \neq 0$, $\phi^t_v \neq 0$ and $\phi^{\partial_1(t)}_w \neq 0$. If $u = \partial_0(s)$ and $w = \partial_1(s)$ then $v = s$; thus, by Proposition 2.4 $\partial_0(t) = \phi_0(\partial_0(s))$ and $\partial_1(t) = \phi_0(\partial_1(s))$. Otherwise $l(v) < l(s)$, and, by induction hypothesis, there exists a decomposition (i, q, j) of v such that x_t occurs in $\phi(x_q)$, $\partial_0(t) = \phi_0(\partial_0(q))$ and $\partial_1(t) = \phi_0(\partial_1(q))$. We obtain the proof of the existence of the decomposition

(p, q, r) of s by:

$$\sum_{a,b}\begin{bmatrix} a \\ u, i\end{bmatrix}\begin{bmatrix} s \\ a, q, b\end{bmatrix}\begin{bmatrix} b \\ j, w\end{bmatrix}=\sum_{c}\begin{bmatrix} s \\ u, c, w\end{bmatrix}\begin{bmatrix} c \\ i, q, j\end{bmatrix}>0. \quad \square$$

Proposition 2.7. *Let $\phi : C(\mathbb{S}) \to C(\mathbb{T})$ be a coalgebra map. If $\mathbb{T}$ is an n-regular hereditarily finite decomposition structure then, for each $m \leqslant n$, $\phi(C(\mathbb{S}_m)) \subseteq C(\mathbb{T}_n)$.*

Proof. The special case $n = 0$ has been considered in Proposition 2.4. Now let us suppose $n > 0$. If $x_s \in K[S_m]$, $m \leqslant n$, then $\bar{\Delta}^n_S(x_s) = 0$. Hence, by Corollary 2.5, $\bar{\Delta}^n_T(\phi(x_s)) = 0$. Thus, owing to the n-regularity of $\mathbb{T}$, $\phi(x_s) \in K[T_n]$. $\square$

Corollary 2.8. *Under the hypotheses of Proposition 2.7, if x_t occurs in $\phi(x_s)$ and $l(t) \geqslant l(s) = n + 1$ then $\partial_0(t) = \phi_0(\partial_0(s))$ and $\partial_1(t) = \phi_0(\partial_1(s))$.*

Proof. Since $l(s)$ is finite and x_t occurs in $\phi(x_s)$, by Proposition 2.6, there exists a decomposition (p, q, r) of s such that x_t occurs in $\phi(x_q)$. Thus, by Proposition 2.7, $l(q) \geqslant n + 1$. Therefore $q = s$ and, by Proposition 2.6, we get the thesis. $\square$

Combining Proposition 2.7 and Corollary 2.8 one can easily prove the following result.

Proposition 2.9. *Let $\phi : C(\mathbb{S}) \to C(\mathbb{T})$ be a coalgebra map. If $\mathbb{T}$ is both an n-regular and $(n + 1)$-regular hereditarily finite decomposition structure then, for each pair $p, q \in S_0$,*

$$\phi(C(\mathbb{S}_{(n+1)}(p, q) \cup \mathbb{S}_n)) \subseteq C(\mathbb{T}_{(n+1)}(\phi_0(p), \phi_0(q)) \cup \mathbb{T}_n).$$

Proof. Since $\mathbb{T}$ is both an n-regular and $(n + 1)$-regular decomposition structure, by Proposition 2.7, $\phi(C(\mathbb{S}_{n+1})) \subseteq C(\mathbb{T}_{n+1})$ and $\phi(C(\mathbb{S}_n)) \subseteq C(\mathbb{T}_n)$. Moreover, by Corollary 2.8, if $s \in S_{(n+1)}(p, q)$ then

$$\phi(x_s) \in C(\mathbb{T}_{(n+1)}(\phi_0(p), \phi_0(q)) \cup \mathbb{T}_n)$$

and the thesis holds. $\square$

Corollary 2.10. *Let $\phi : C(\mathbb{S}) \to C(\mathbb{T})$ be a coalgebra isomorphism. If $\mathbb{S}$, $\mathbb{T}$ are both n-regular and $(n + 1)$-regular hereditarily finite decomposition structures then, for every pair $p, q \in S_0$, the cardinality of $S_{(n+1)}(p, q)$ is equal to the cardinality of $T_{(n+1)}(\phi_0(p), \phi_0(q))$.*

Proof. By Proposition 2.9, $C(\mathbb{S}_{(n+1)}(p, q) \cup \mathbb{S}_n) \simeq C(\mathbb{T}_{(n+1)}(\phi_0(p), \phi_0(q)) \cup \mathbb{T}_n)$. By Proposition 2.7, $C(\mathbb{S}_n) \simeq C(\mathbb{T}_n)$. Hence the K-spaces

$$K[S_{(n+1)}(p, q)] \simeq K[S_{(n+1)}(p, q) \cup S_n]/K[S_n]$$

and

$$K[T_{(n+1)}(\phi_0(p),\ \phi_0(q))] \simeq K[T_{(n+1)}(\phi_0(p),\ \phi_0(q)) \cup T_n]/K[T_n]$$

have the same dimension. $\square$

Now we are able to state the main result of this section.

Proposition 2.11. *Let $\mathbb{S}$, $\mathbb{T}$ be a regular hereditarily finite decomposition structures. If $C(\mathbb{S})$ and $C(\mathbb{T})$ are isomorphic coalgebras then $\mathbb{S}$ and $\mathbb{T}$ have isomorphic presentations.*

We observe that the hypotheses of n-regularity and $(n+1)$-regularity are necessary in Corollary 2.10 as the following example shows.

Let $S = \{u_1, u_2, u_3, p, q, r, s\}$. If $S_0 = \{u_1, u_2, u_3\}$, $S_{(1)} = \{p, q, r\}$, $S_{(2)} = \{s\}$ with $u_1 = \partial_0(p) = \partial_0(r) = \partial_0(s)$, $u_2 = \partial_1(p) = \partial_0(q)$, $u_3 = \partial_1(q) = \partial_1(r) = \partial_1(s)$ and $[{}_{p,q}^{\ s}] = 2$ then $\mathbb{S}$ is a regular decomposition structure. While, if $T = \{u_1', u_2', u_3', p', q', r', s'\}$, $T_0 = \{u_1', u_2', u_3'\}$, $T_{(1)} = \{p', q'\}$, $T_{(2)} = \{r', s'\}$, $u_1' = \partial_0(p') = \partial_0(r') = \partial_0(s')$, $u_2' = \partial_1(p') = \partial_0(q')$, $u_3' = \partial_1(q') = \partial_1(r') = \partial_1(s')$ and $[{}_{p',q'}^{\ s'}] = [{}_{p',q'}^{\ r'}] = 1$ then $\mathbb{T}$ is not a 1-regular decomposition structure. For $x_{s'} - x_{r'} \in \operatorname{Ker}(\bar{\delta}_T^1)$ and $x_{s'} - x_{r'} \notin \mathbb{Z}[T_1]$. Obviously $\mathbb{S}$ and $\mathbb{T}$ do not have isomorphic presentations while the linear map $\phi : C(\mathbb{S}) \to C(\mathbb{T})$ defined by $\phi(x_{u_i}) = x_{u_i'}$, $i = 1, 2, 3$, $\phi(x_p) = x_{p'}$, $\phi(x_q) = x_{q'}$, $\phi(x_s) = x_{s'} + x_{r'}$ and $\phi(x_r) = x_{s'} - x_{r'}$ is a coalgebra isomorphism.

3. Incidence algebras

Let R be a commutative ring and A, B two R-modules. A family $(f_i)_{i\in I}$ of elements of $\operatorname{Hom}_R(A, B)$ is said to be a *summable family* if, for every $a \in A$, the set $\{i \in I \mid f_i(a) \neq 0\}$ is finite. Given a summable family $(f_i)_{i\in I}$ of elements of $\operatorname{Hom}_R(A, B)$ we obtain a new element $\sum_{i\in I} f_i$ of $\operatorname{Hom}_R(A, B)$ putting, for each $a \in A$,

$$\Big(\sum_{i\in I} f_i\Big)(a) := \sum_{i\in I} f_i(a).$$

Thus, if $A = R[S]$ and $B = R$ the set $(x^s)_{s\in S}$, where

$$x^s : R[S] \longrightarrow R$$
$$x_r \mapsto \begin{cases} 1 & \text{if } r = s \\ 0 & \text{otherwise} \end{cases}$$

is a summable family. Obviously, if f is an arbitrary element of $R[S]^* = \operatorname{Hom}_R(R[S], R)$ then the family $(f(x_s)x^s)_{s\in S}$ is a summable family and we have:

$$\sum_{s\in S} f(x_s)x^s = f.$$

Proposition 3.1. *Let $\psi : R[T]^* \to R[S]^*$ be a linear map. Then the following statements are equivalent*:

(i) *there exists a linear map $\phi : R[S] \to R[T]$ such that $\psi(f) = f \circ \phi$ (i.e. ψ is the dual map of ϕ)*;

(ii) *if the family $(f_i)_{i \in I}$ is a summable family, then the family $(\psi(f_i))_{i \in I}$ is also a summable family and $\psi(\sum_{i \in I} f_i) = \sum_{i \in I} \psi(f_i)$*;

(iii) *the family $(\psi(x^t))_{t \in T}$ is a summable family and, for each $f \in R[T]^*$, $\psi(f) = \sum_{t \in T} f(x_t)\psi(x^t)$.*

If we consider R as a topological ring with the discrete topology then $R[S]^*$ is naturally provided with a structure of topological module by the product topology (or finite topology). According to this scheme of things Proposition 3.1 can be restated in the following way:

Proposition 3.2. *A linear map $\psi : R[T]^* \to R[S]^*$ is continuous, with respect to finite topology, if and only if ψ is the dual map of a linear map $\phi : R[S] \to R[T]$.*

We shall assume throughout that R is a characteristic zero field K. We recall that the *Incidence Algebra* of a decomposition structure $\mathbb{S}$ is the dual algebra of the Incidence Coalgebra $C(\mathbb{S})$. Thus, the product $f * g$ of the elements $f, g \in K[S]^*$ is given by

$$(f * g)(x_s) := m \circ (f \otimes g) \circ \Delta_S(x_s) = \sum_{q,r} \begin{bmatrix} s \\ q, r \end{bmatrix} f(x_q) g(x_r),$$

where $m : K \otimes K \to K$ is the product over K.

It is plain that this product is associative and the linear map ε_S is the two-sided identity. The Incidence Algebra of $\mathbb{S}$ will be denoted $A(\mathbb{S})$.

We begin our study of the algebra $A(\mathbb{S})$ by considering two families of its ideals.

Let X be a subset of S, denoted by $d(X)$ the set of the elements of S which are in at least one decomposition of an element of X, then

$$d(X)^{\perp} := \{f \in A(\mathbb{S}) \mid f(x_s) = 0 \quad \text{for every} \quad s \in d(X)\}$$

is a two-sided ideal of $A(\mathbb{S})$.

Another family of two-sided ideals of $A(\mathbb{S})$, which we need to consider is, for each $n \in \mathbb{N}$,

$$J_n(\mathbb{S}) := \{f \in A(\mathbb{S}) \mid s \in S_{n-1} \Rightarrow f(x_s) = 0\}$$

Moreover we have:

$$A(\mathbb{S}) = J_0(\mathbb{S}) \supseteq J_1(\mathbb{S}) \supseteq \cdots \quad \text{and} \quad (J_1(\mathbb{S}))^n \subseteq J_n(\mathbb{S}).$$

It is easy to check that the quotient algebra $A(\mathbb{S})/J_1(\mathbb{S})$ is isomorphic to the algebra $A(\mathbb{S}_0)$ obtained providing $K[S_0]^*$ with the product

$$(f * g)(x_s) := f(x_s)g(x_s).$$

We observe that the family $(x^s)_{s\in S_0}$ is the unique maximal family of non-zero orthogonal primitive idempotent elements of $A(\mathbb{S}_0)$.

The existence of the ideals $d(X)^\perp$ and $J_n(\mathbb{S})$ as well as of the algebra $A(\mathbb{S}_0)$ allows us to state that if we consider K provided with the discrete topology and $A(\mathbb{S})$ with the product topology then $A(\mathbb{S})$ is an *Abstract Incidence Algebra* over K according to the definition given by Dür in [5]. In this case $H^{(0)} = A(\mathbb{S}_0)$, $H_n = J_n(\mathbb{S})$ and we obtain a 0-basis of two-sided ideals considering the family $d(X)^\perp$ when X is a finite subset of S.

If $\mathbb{S}$ is a hereditarily finite decomposition structure and $f \in J_1(\mathbb{S})$ then the family $(f^n)_{n\in\mathbb{N}}$ is a summable family. As a consequence it is possible to prove (see [7]) the following statement about the invertible elements of $A(\mathbb{S})$.

Proposition 3.3. *If $\mathbb{S}$ is a hereditarily finite decomposition structure then an element $f \in A(\mathbb{S})$ is invertible if and only if $f(x_s)$ is an invertible element of K for each $s \in S_0$.*

Corollary 3.4. *$J_1(\mathbb{S})$ is the Jacobson radical of $A(\mathbb{S})$.*

For our purposes the knowledge of the inter-relation between the ideals $J_n(\mathbb{S})$ and $J_1(\mathbb{S}))^n$ is fundamental. We already know that $(J_1(\mathbb{S}))^n \subseteq J_n(\mathbb{S})$ but, generally, $J_n(S) \nsubseteq (J_1(S))^n$. That is especially evident when $\mathbb{S}$ is not an n-regular decomposition structure. In fact, the following proposition shows that the set $\{f \in J_{n+1}(\mathbb{S}) \mid f(x_s) = 0$ for all but a finite number of $s \in S\}$ is not contained in $(J_1(\mathbb{S}))^{n+1}$ if $\mathbb{S}$ is not an n-regular decomposition structure.

Proposition 3.5. *The following statements are equivalent*:

(i) *$\mathbb{S}$ is an n-regular decomposition structure*;

(ii) *for every pair of finite sequences $s_1, \ldots, s_m \in S$, $l(s_i) > n$, and $k_1, \ldots, k_m \in K$ there exists a finite family of scalars $h_{r_1\cdots r_{n+1}}$, with $(r_1, \ldots, r_{n+1})$ proper decomposition of degree $n+1$ of s_i, $1 \leq i \leq m$, such that $h(x_{s_i}) = k_i$, where $h = \sum h_{r_1\cdots r_{n+1}} x^{r_1} * \cdots * x^{r_{n+1}}$*;

(iii) *for every pair of finite sequences $s_1, \ldots, s_m \in S$, $l(s_i) > n$, and $k_1, \ldots, k_m \in K$ there exists $f \in (J_1(S))^{n+1}$ such that $f(x_{s_i}) = k_i$, $1 \leq i \leq m$.*

Proof. (i)$\Rightarrow$(ii). If $\mathbb{S}$ is an n-regular decomposition structure, then the K-space spanned by $\bar{\Delta}^n_S(x_{s_i})$'s, $l(s_i) > n$ and $1 \leq i \leq m$, does not have dimension less than m. Hence, if $\bar{\Delta}^n_S(x_{s_i}) = \sum \left[{}_{r_1\cdots r_{n+1}}^{\;s_i}\right] x_{r_1} \otimes \cdots \otimes x_{r_{n+1}}$, there exist $h_{r_1\cdots r_{n+1}}$ such that

$\sum \left[\begin{smallmatrix} s_i \\ r_1\cdots r_{n+1}\end{smallmatrix}\right] h_{r_1\cdots r_{n+1}} = k_i$. Thus

$$h = \sum h_{r_1\cdots r_{n+1}} x^{r_1} * \cdots * x^{r_{n+1}}$$

satisfies the conditions $h(x_{s_i}) = k_i$.

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). If $\mathbb{S}$ is not an n-regular decomposition structure there exist a finite sequence $s_1, \ldots, s_m$ of elements of S, $l(s_i) > n$, and a finite sequence $h_1, \ldots, h_m$ of elements of K such that $\sum_{i=1}^m h_i \bar{\Delta}_S^n(x_{s_i}) = 0$. Thus $\sum_{i=1}^m h_i \left[\begin{smallmatrix} s_i \\ r_1\cdots r_{n+1}\end{smallmatrix}\right] = 0$ for every $r_1, \ldots, r_{n+1} \in S - S_0$. Now, if $g = g_1 * \cdots * g_{n+1}$, $g_j \in J_1(\mathbb{S})$, $1 \leq j \leq n+1$, then

$$\begin{aligned} \sum_{i=1}^m h_i g(x_{s_i}) &= \sum_{i=1}^m h_i \sum \begin{bmatrix} s_i \\ r_1 \cdots r_{n+1}\end{bmatrix} g_1(x_{r_1}) \cdots g_{n+1}(x_{r_{n+1}}) \\ &= \sum \left(\sum_{i=1}^m h_i \begin{bmatrix} s_i \\ r_1 \cdots r_{n+1}\end{bmatrix}\right) g_1(x_{r_1}) \cdots g_{n+1}(x_{r_{n+1}}) = 0. \end{aligned}$$

So, for every $f \in (J_1(\mathbb{S}))^{n+1}$, we have $\sum_{i=1}^m h_i f(x_{s_i}) = 0$. Therefore, if $k_1, \ldots, k_m$ is a sequence of elements of K with $\sum_{i=1}^m h_i k_i \neq 0$ there exists no element $f \in (J_1(\mathbb{S}))^{n+1}$ such that $f(x_{s_i}) = k_i$, $1 \leq i \leq m$. □

As a consequence of the former statement we see that if the set $\{s_i \in S \mid l(s_i) > n\}$ is finite and $\mathbb{S}$ is an n-regular decomposition structure then $(J_1(\mathbb{S}))^n = J_n(\mathbb{S})$. However, also under the hypothesis of n-regularity for $\mathbb{S}$, generally, $(J_1(\mathbb{S}))^n \neq J_n(\mathbb{S})$. The only thing that we can state, by Proposition 3.5, is that if we consider $A(\mathbb{S})$ equipped with the finite topology then $\mathbb{S}$ is an n-regular decomposition structure if and only if $J_n(\mathbb{S})$ is the topological closure of $(J_1(\mathbb{S}))^n$. Nevertheless, if $\mathbb{S}$ is a finitely generated decomposition structure we can prove a "local" equality between $(J_1(\mathbb{S}))^n$ and $J_n(\mathbb{S})$.

Proposition 3.6. *If $\mathbb{S}$ is a finitely generated decomposition structure, then, for each $n \geq 1$ and for each pair $p, q \in S_0$,*

$$x^p * J_n(\mathbb{S}) * x^p = x^r * (J_1(\mathbb{S}))^n * x^q.$$

Proof. It is sufficient to prove that $x^p * J_n(\mathbb{S}) * x^q \subseteq x^p * (J_1(\mathbb{S}))^n * x^q$. The proof is by induction on n. The proposition clearly holds when $n = 1$. Thus, we suppose the conclusion holds for n. By Proposition 1.10, since $\mathbb{S}$ is a finitely generated decomposition structure, the sets $S_{(m)}(p, q)$ are finite for each $m \geq 1$. Suppose $f \in x^p * J_{n+1}(\mathbb{S}) * x^q$. By Proposition 3.5, we can find a sequence of linear forms

$g_{n+i} \in (J_1(\mathbb{S}))^{n+i}$, $i = 1, 2, \ldots$, such that

$$g_{n+1}(x_r) = f(x_r) \quad \text{for every } r \in S_{(n+1)}(p, q)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$g_{n+i}(x_r) = f(x_r) - \sum_{j=1}^{i-1} g_{n+j}(x_r) \quad \text{for every } r \in S_{(n+i)}(p, q)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where $g_{n+i} = \sum h_{r_1 \cdots r_{n+i}} x^{r_1} * \cdots * x^{r_{n+i}}$ and the sum ranges over all the proper decompositions of degree $n + i$ of the elements of $S_{(n+i)}(p, q)$. Obviously, the family $(g_{n+i})_{i\geq 1}$ is a summable family and we have $f = \sum_{i\geq 1} g_{n+i}$. Therefore:

$$f = \sum_{\substack{u\in[p,q] \\ v\in S_{(1)}(p,u)}} \left(x^{v} * \left(\sum h_{r_1 \cdots r_{n+i}} x^{r_2} * \cdots * x^{r_{n+i}}\right)\right)$$

where the second sum ranges over all the proper decompositions $(r_1, \ldots, r_{n+i})$ of the elements of $\bigcup_{i\geq 1} S_{(n+i)}(p, q)$ with $r_1 = v$. Thus

$$f = \sum_{\substack{u\in[p,q] \\ v\in S_{(1)}(p,u)}} x^{v} * g_{v,v}$$

with $g_{v,v} \in x^{\partial_1(v)} * J_n(\mathbb{S}) * x^q$. So, by induction hypothesis, and, since the set of pairs (u, v) such that $u \in [p, q]$ and $v \in S_{(1)}(p, u)$ is finite, we can conclude that $f \in x^p * (J_1(\mathbb{S}))^{n+1} * x^q$. $\square$

We now come to the study of the algebra maps between two Incidence Algebras and their relationships with the coalgebra maps. With reference to this we observe that to be sure that an algebra map ψ is the dual of a coalgebra map it is sufficient to make sure that ψ is the dual of a linear map. In fact, the following proposition holds.

Proposition 3.7. *If the algebra map $\psi: A(\mathbb{T}) \to A(\mathbb{S})$ is dual of a linear map $\phi: C(\mathbb{S}) \to C(\mathbb{T})$ then ϕ is a coalgebra map.*

Consequently, by Proposition 3.1, we have;

Proposition 3.8. *Let $\psi: A(\mathbb{T}) \to A(\mathbb{S})$ be an algebra map. If*
(i) *the family $(\psi(x^t))_{t\in T}$ is a summable family;*
(ii) *for each family $(k_t)_{t\in T}$ of elements of K, $\psi(\sum_{t\in T} k_t x^t) = \sum_{t\in T} k_t \psi(x^t)$;*
then ψ is dual of a coalgebra map $\phi: C(\mathbb{S}) \to C(\mathbb{T})$.

In the following, owing to Proposition 3.2, if the algebra map ψ satisfies the hypotheses of Proposition 3.8 we shall say that ψ is a *continuous algebra map.*

Corollary 3.9. *Every inner automorphism of $A(\mathbb{S})$ is a continuous automorphism.*

We begin our study of the algebra maps with a simple proposition. This result has been proved by Leroux (see [8]) in the particular case of the Möbius Categories, but it is possible to repeat the same proof in our case.

Proposition 3.10. *Let $\mathbb{T}$ be a hereditarily finite decomposition structure. If $\psi : A(\mathbb{T}) \to A(\mathbb{S})$ is an algebra map then*

$$\psi(J_1(\mathbb{T})) \subseteq J_1(\mathbb{S}).$$

Proposition 3.11. *Let $\mathbb{T}$ be a hereditarily finite decomposition structure and $\psi : A(\mathbb{T}) \to A(\mathbb{S})$ an algebra map. If $l(s)$ is finite and if $\psi(x^t)(x_s) \neq 0$ then there exists a decomposition (u, v, w) of s such that $\psi(x^t)(x_v) \neq 0$ and $\psi(x^{\partial_0(t)})(x_{\partial_0(v)}) = \psi(x^{\partial_1(t)}(x_{\partial_1(v)}) = 1$.*

Proof. If $l(s) = 0$ then $s = \partial_0(s) = \partial_1(s)$. From $\psi(x^t)(x_s) \neq 0$ we get:

$$\psi(x^{\partial_0(t)} * x^t * x^{\partial_1(t)})(x_s) = \psi(x^{\partial_0(t)})(x_s)\psi(x^t)(x_s)\psi(x^{\partial_1(t)})(x_s) \neq 0.$$

So, $\psi(x^{\partial_0(t)})(x_s) = k \neq 0$ and, since

$$\psi(x^{\partial_0(t)})(x_s) = \psi(x^{\partial_0(t)})(x_s)\psi(x^{\partial_0(t)})(x_s),$$

we have $k = 1$. Let us suppose, now, that the proposition holds for every element of S with length less than m. If $l(s) = m$ we have:

$$\psi(x^t)(x_s) = \sum_{a,b,c} \begin{bmatrix} & s & \\ a, & b, & c \end{bmatrix} \psi(x^{\partial_0(t)})(x_a)\psi(x^t)(x_b)\psi(x^{\partial_1(t)})(x_c) \neq 0.$$

Consequently, there exists a decomposition (a, b, c) of s such that $\psi(x^{\partial_0(t)})(x_a) \neq 0$, $\psi(x^t)(x_b) \neq 0$, $\psi(x^{\partial_1(t)})(x_c) \neq 0$. If $a = \partial_0(s)$ and $c = \partial_1(s)$ then $b = s$ and the proof is concluded. Otherwise $l(b) < m$ and, by induction hypothesis, there exists a decomposition (q, v, r) of b such that

$$\psi(x^{\partial_0(t)})(x_{\partial_0(v)}) = \psi(x^{\partial_1(t)})(x_{\partial_1(v)}) = 1 \quad \text{and} \quad \psi(x^t)(x_v) \neq 0$$

Now, as in Proposition 2.6, we have:

$$\sum_{u,w} \begin{bmatrix} u \\ a, \; q \end{bmatrix} \begin{bmatrix} & s & \\ u, & v, & w \end{bmatrix} \begin{bmatrix} w \\ r, \; c \end{bmatrix} = \sum_p \begin{bmatrix} & s & \\ a, & p, & c \end{bmatrix} \begin{bmatrix} & p & \\ q, & v, & r \end{bmatrix} > 0.$$

Consequently there exists a decomposition (u, v, w) of s such that

$$\psi(x^{\partial_0(t)})(x_{\partial_0(v)}) = \psi(x^{\partial_1(t)})(x_{\partial_1(v)}) = 1 \quad \text{and} \quad \psi(x^t)(x_v) \neq 0. \quad \Box$$

Corollary 3.12. *Let $\mathbb{S}$, $\mathbb{T}$ be hereditarily finite decomposition structures. If $\psi : A(\mathbb{T}) \to A(\mathbb{S})$ is an isomorphism then, for each $s \in S_0$, there exists $t \in T_0$ such that $\psi(x^t)(x_s) = 1$.*

Proof. Since ψ is onto, for every $s \in S_0$, there exists an element $f = \sum_{t \in T} f_t x^t$ of $A(\mathbb{T})$ such that $\psi(f) = x^s$. Let $X = \{t \in T \mid f_t \neq 0\}$; by Proposition 3.10 $f \notin J_1(\mathbb{T})$, therefore $X \cap T_0 \neq \emptyset$. If $t \in X \cap T_0$ then $x^t * f = x^t + \sum_{v \in Y} f_v x^v$ where $Y = \{v \in X \mid l(v) > 0 \text{ and } \partial_0(v) = t\}$. Thus

$$\psi(x^t) + \psi\Big(\sum_{v \in Y} f_v x^v\Big) = \psi(x^t) * x^s.$$

Since $\psi(\sum_{v \in Y} f_v x^v) \in J_1(\mathbb{S})$, we find that if $r \in S_0$ and $r \neq s$ then $\psi(x^t)(x_r) = 0$. But $\psi(x^t) \neq 0$, therefore there exists $w \in S$ such that $\psi(x^t)(x_w) \neq 0$. So, by Proposition 3.11, $\psi(x^t)(x_s) = 1$. □

Proposition 3.13. *Let $\mathbb{S}$, $\mathbb{T}$ be hereditarily finite decomposition structures and $\psi : A(\mathbb{T}) \to A(\mathbb{S})$ an algebra map. If, for every $s \in S_0$, there exists $t \in T_0$ such that $\psi(x^t)(x_s) = 1$ then there exists an inner automorphism ι of $A(\mathbb{S})$ such that, for every $q \in T_0$ $(\iota \circ \psi)(x^q)(x_r) = 0$ whenever $l(r) > 0$.*

Proof. At first we observe that if $s \in S_0$ and there exists at most one $t \in T_0$ such that $\psi(x^t)(x_s) = 1$. In fact, if $u \in T_0$, $u \neq t$, and $\psi(x^u)(x_s) = 1$ then $1 = \psi(x^t)(x_s)\psi(x^u)(x_s) = \psi(x^t * x^u)(x_s) = 0$. Consequently, by hypothesis, for each $s \in S_0$ there exists a unique $t_s \in T_0$ such that $\psi(x^{t_s})(x_s) = 1$. Since $(\psi(x^{t_s}) * x^s)_{s \in S_0}$ is a summable family of $A(\mathbb{S})$ putting $g = \sum_{s \in S_0} \psi(x^{t_s}) * x^s$ we obtain an invertible element of $A(\mathbb{S})$. If $t \in T_0$, then

$$\psi(x^t) * g = \sum_{s \in S_0} \psi(x^t) * \psi(x^{t_s}) * x^s = \sum_{\{s \in S_0 \mid t_s = t\}} \psi(x^{t_s}) * x^s$$

$$= \sum_{\{s \in S_0 \mid t_s = t\}} g * x^s = g * \sum_{\{s \in S_0 \mid t_s = t\}} x^s.$$

Therefore

$$g^{-1} * \psi(x^t) * g = \sum_{\{s \in S_0 \mid t_s = t\}} x^s.$$

Thus the map

$$\begin{aligned} \iota : A(\mathbb{S}) &\longrightarrow A(\mathbb{S}) \\ h &\mapsto g^{-1} * h * g \end{aligned}$$

satisfies our requirements. □

If we limit ourselves to consider Incidence Algebras of finitely generated decomposition structures it is possible to give more detailed results. Particularly, under this hypothesis, it is possible to find conditions such that an algebra map is continuous and to prove a proposition analogous to Proposition 2.11.

Proposition 3.14. *Let $\mathbb{T}$ be a finitely generated decomposition structure. If $\psi : A(\mathbb{T}) \to A(\mathbb{S})$ is an algebra map then, for each $u, v \in T_0$,*

$$\psi(x^u * J_n(\mathbb{T}) * x^v) \subseteq (J_1(\mathbb{S}))^n.$$

Proof. If $u, v \in T_0$ then, by Propositions 3.6 and 3.10, we have:

$$\begin{aligned}\psi(x^u * J_n(\mathbb{T}) * x^v) &= \psi(x^u * (J_1(\mathbb{T}))^n * x^v) \subseteq \psi(x^u) * (J_1(\mathbb{S}))^n * \psi(x^v)\\ &= (J_1(\mathbb{S}))^n. \quad \square\end{aligned}$$

Corollary 3.15. *If $t \in T$, $s \in S$ and $\psi(x^t)(x_s) \neq 0$ then $l(t) \leq l(s)$.*

Proof. If $l(t) = n$ then $x^t \in x^{\partial_0(t)} * J_n(\mathbb{T}) * x^{\partial_1(t)}$. So, by the proposition, if $l(s) < l(t)$ then $\psi(x^t)(x_s) = 0$. $\square$.

Corollary 3.16. *If $\mathbb{S}$ is hereditarily finite, then the family $(\psi(x^t))_{t\in T}$ is a summable family of elements of $A(\mathbb{S})$.*

Proof. Let $T_s = \{t \in T \mid \psi(x^t)(x_s) \neq 0\}$ and let $T_{0s} = \{(p, q) \mid p = \partial_0(t),\ q = \partial_1(t)$ and $t \in T_s\}$. If $s \in S_m$, since the number of decompositions (u, v, w) of s is finite, by Proposition 3.11, we see that T_{0s} is finite. Now, by Corollary 3.15, $T_s \subseteq \bigcup_{(p,q)\in T_{0s}} T_m(p, q)$; there fore, by Proposition 1.10, we can conclude that T_s is finite. $\square$

Proposition 3.17. *Let $\mathbb{T}$ be a finitely generated decomposition structure and let $\mathbb{S}$ be a hereditarily finite decomposition structure. An algebra map $\psi: A(\mathbb{T}) \to A(\mathbb{S})$ is continuous if and only if, for each $s \in S_0$, there exists $t \in T_0$ such that $\psi(x^t)(x_s) = 1$.*

Proof. If $\psi: A(\mathbb{T}) \to A(\mathbb{S})$ is dual of a coalgebra map $\phi: C(\mathbb{S}) \to C(\mathbb{T})$ then $\psi(x^t) = x^t \circ \phi$. Now, by Proposition 2.4, for every $s \in S_0$ there exists $t \in T_0$ such that $\phi(x_s) = x_t$. So, for every $s \in S_0$ there exists $t \in T_0$ such that $\psi(x^t)(x_s) = (x^t \circ \phi)(x_s) = x^t(x_t) = 1$.

Vice versa let us suppose that, for each $s \in S_0$, there exists $t \in T_0$ such that $\psi(x^t)(x_s) = 1$. In this case, by Proposition 3.13, there exists an inner automorphism ι of $A(\mathbb{S})$ such that, for each $q \in T_0$, $(\iota \circ \psi)(x^q)(x_r) = 0$ whenever $l(r) > 0$. Now, by Corollary 3.9, ι is continuous; so to conclude the proof, it is sufficient to prove that $\iota \circ \psi$ is continuous. By Corollary 3.16, $((\iota \circ \psi)(x^t))_{t\in T}$ is a summable family. Therefore we must only prove that, for each family $(k_t)_{t\in T}$ of elements of K and for each $w \in S$, we have:

$$(\iota \circ \psi)\Big(\sum_{t\in T} k_t x^t\Big)(x_w) = \sum_{t\in T} k_t (\iota \circ \psi)(x^t)(x_w).$$

Let $w \in S_m$. If we put $\partial_0(w) = u$, $\partial_1(w) = v$ and if we denote t_u, t_v the elements of T_0 such that $\psi(x^{t_u})(x_u) = \psi(x^{t_v})(x_v) = 1$ then we have:

$$\begin{aligned}(\iota \circ \psi)\Big(\sum_{t\in T} k_t x^t\Big)(x_w) &= \Big(x^u * (\iota \circ \psi)\Big(\sum_{t\in T} k_t x^t\Big) * x^v\Big)(x_w)\\ &= (\iota \circ \psi)\Big(x^{t_u} * \sum_{t\in T} k_t x^t * x^{t_v}\Big)(x_w) = (\iota \circ \psi)\Big(\sum_{t\in T(t_u,t_v)} k_t x^t\Big)(x_w)\\ &= (\iota \circ \psi)\Big(\sum_{t\in T_m(t_u,t_v)} k_t x^t\Big)(x_w) + (\iota \circ \psi)\Big(\sum_{t\in (T-T_m)(t_u,t_v)} k_t x^t\Big)(x_w).\end{aligned}$$

Now

$$\sum_{t\in(T-T_m)(t_u,t_v)} k_t x^t \in x^{t_u} * J_{m+1}(\mathbb{T}) * x^{t_v};$$

thus, since $T_m(t_u, t_v)$ is finite, we have:

$$(\iota\circ\psi)\Big(\sum_{t\in T} k_t x^t\Big)(x_w) = \sum_{t\in T_m(t_u,t_v)} k_t(\iota\circ\psi)(x^t)(x_w) = \sum_{t\in T} k_t(\iota\circ\psi)(x^t)(x_w). \quad \square$$

As a consequence of the former statement we obtain a result which generalizes Leroux's proposition about the Isomorphism Problem for Incidence Algebras of Möbius Categories (see [8]).

Corollary 3.18. *Let $\mathbb{S}$, $\mathbb{T}$ be finitely generated decomposition structures. Every algebra isomorphism $\psi : A(\mathbb{T}) \to A(\mathbb{S})$ is continuous.*

Proof. By Corollary 3.12 and Proposition 3.17. $\square$

Corollary 3.19. *Let $\mathbb{S}$, $\mathbb{T}$ be finitely generated decomposition structures. If $A(\mathbb{S})$ and $A(\mathbb{T})$ are isomorphic Incidence Algebras then $\mathbb{S}$ and $\mathbb{T}$ have isomorphic presentations.*

Proof. If $A(\mathbb{T}) \simeq A(\mathbb{S})$ then, by Corollary 3.18, $C(\mathbb{S}) \simeq C(\mathbb{T})$. Thus, by Proposition 2.11, we come to the conclusion. $\square$

Proposition 3.17 shows that if $\mathbb{T}$ is a finitely generated decomposition structure then the occurrence that ψ is continuous results from its behaviour on the set T_0. This is made more evident by the following proposition.

Proposition 3.20. *Let $\mathbb{T}$ be a finitely generated decomposition structure and let $\mathbb{S}$ be a hereditarily finite decomposition structure. An algebra map $\psi : A(\mathbb{T}) \to A(\mathbb{S})$ is continuous if and only if is continuous the algebra map*

$$\begin{aligned} \psi : A(\mathbb{T})/J_1(\mathbb{T}) &\longrightarrow A(\mathbb{S})/J_1(\mathbb{S}) \\ f + J_1(\mathbb{T}) &\mapsto \psi(f) + J_1(\mathbb{S}). \end{aligned}$$

Therefore, now we can restrict our analysis, about the relationships between algebra and coalgebra maps, only to include the algebra maps between $A(\mathbb{T}_0)$ and $A(\mathbb{S}_0)$. The main instrument, which we shall use, is the notion of ultrafilter in the lattice $\mathbb{P}(T_0)$ of all the subsets of T_0. This is related to the following proposition.

Proposition 3.21. *Let $\psi : A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ an algebra map. Then for each $s \in S_0$ the set*

$$\mathbb{F}_s = \Big\{X \subseteq T_0 \mid \psi\Big(\sum_{t\in X} x^t\Big)(x_s) = 1\Big\}$$

is an ultrafilter in $\mathbb{P}(T_0)$.

Therefore the following proposition holds:

Proposition 3.22. *An algebra map $\psi: A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ is continuous if and only if, for each $s \in S_0$, $\mathbb{F}_s$ is a principal ultrafilter in $\mathbb{P}(T_0)$.*

Corollary 3.18 assures us that every isomorphism between Incidence Algebras of finitely generated decomposition structures is always dual of a coalgebra map. The following propositions establish conditions on T_0 in order that every algebra map $\psi: A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ is dual of a coalgebra map.

Proposition 3.23. *Let $\psi: A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ be an algebra map. Then $\psi(\sum_{t \in T_0} k_t x^t)(x_s) = 0$ if and only if $\{t \in T_0 \mid k_t = 0\} \in \mathbb{F}_s$.*

Proof. Let us suppose $\psi(\sum_{t \in T_0} k_t x^t)(x_s) = 0$. If $X = \{t \in T_0 \mid k_t = 0\} \notin \mathbb{F}_s$ then, putting $X' = \{t \in T_0 \mid k_t \neq 0\}$, we have:

$$\psi\Big(\sum_{t \in X'} k_t x^t\Big)(x_s)\psi\Big(\sum_{t \in X'} k_t^{-1} x^t\Big)(x_s) = \psi\Big(\sum_{t \in X'} x^t\Big)(x_s) = 1, \text{ since } X' \in \mathbb{F}_s.$$

Therefore $\psi(\sum_{t \in T_0} k_t x^t)(x_s) = \psi(\sum_{t \in X'} k_t x^t)(x_s) \neq 0$. Vice versa suppose $X \in \mathbb{F}_s$. Then $\psi(\sum_{t \in T_0} k_t x^t)(x_s) = \psi(\sum_{t \in X'} k_t x^t)(x_s)\psi(\sum_{t \in X} x^t)(x_s) = 0$. □

We recall (see [2]) that, given an arbitrary infinite cardinal α, an ultrafilter $\mathbb{F}$ is said to be α-complete if and only if the intersection of any set of fewer than α elements of $\mathbb{F}$ belongs to $\mathbb{F}$. The α-complete ultrafilters are characterized by the following lemma.

Lemma 3.24. *Let I be a set and let $\mathbb{F}$ be an ultrafilter in $\mathbb{P}(I)$. $\mathbb{F}$ is α-complete if and only if for every partition of I into fewer than α parts, one of the parts belongs to $\mathbb{F}$.*

Proof. See [2] p. 180. □

Proposition 3.25. *Every algebra map $\psi: A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ is continuous if and only if there exists no $|K|^+$-complete non-principal ultrafilter in $\mathbb{P}(T_0)$. ($|K|^+$ denotes the least cardinal greater than the cardinal $|K|$ of the field K).*

Proof. Let $\mathbb{F}$ be a $|K|^+$-complete non principal ultrafilter in $\mathbb{P}(T_0)$. By Lemma 3.24, for each map $f: T_0 \to K$ there exist $X_f \in \mathbb{F}$ and $k_f \in K$ such that $f(t) = k_f$ for each $t \in X_f$. Now, let s be a fixed element of S_0. Then the map $\psi: A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ defined by

$$\psi\Big(\sum_{t \in T_0} f(t) x^t\Big)(x_u) = \begin{cases} k_f & \text{if } u = s \\ 0 & \text{otherwise} \end{cases}$$

is an algebra map. Moreover, since for each $t \in T_0$ $\{t\} \notin \mathbb{F}$, $\psi(x^t)(x_s) = 0$. Thus

$$\psi\Big(\sum_{t\in T} x^t\Big) = \psi(\varepsilon_{T_0}) = \varepsilon_{S_0} \neq \sum_{t\in T_0} \psi(x^t).$$

Conversely suppose that $\psi: A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ is not a dual map. Thus, there exists $s \in S_0$ such that $\mathbb{F}_s$ is not principal. Let $(X_h)_{h\in H}$, $H \subseteq K$, be a partition of T_0 and let $f: T_0 \to K$ be the map defined by $f(t) = h$ whenever $t \in X_h$. We have: if $\psi(\sum_{t\in T_0} f(t)x^t)(x_s) = k$, then $X_k \in \mathbb{F}_s$ and $X_h \notin \mathbb{F}_s$ for each $h \neq k$. In fact

$$\psi\Big(\sum_{t\in T_0} [f(t) - k]x^t\Big)(x_s) = \psi\Big(\sum_{t\in T_0} f(t)x^t\Big)(x_s) - k\psi\Big(\sum_{t\in T_0} x^t\Big)(x_s) = 0.$$

Therefore, by Proposition 3.23, $X_k \in \mathbb{F}_s$ and, by Lemma 3.24, $\mathbb{F}_s$ is $|K|^+$-complete. □

We conclude our observations about the relationships between algebra and coalgebra maps with a proposition which specifies the links that must intervene between $|T_0|$ and $|K|$ so that every algebra map $\psi: A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ is dual of a coalgebra map. If $|T_0|$ is a measurable cardinal (i.e. there exists a non principal $|T_0|$-complete ultrafilter in $\mathbb{P}(T_0)$), then every algebra map $\psi: A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ is continuous if and only if $|T_0| \leq |K|$; while, if $|T_0|$ is not a measurable cardinal then it is sufficient that $|T_0| \leq |K|^+$. In any case, if there exists a cardinal α such that $\alpha < |T_0| \leq 2^\alpha$ then the condition $|K| \geq \alpha$ is sufficient so that every algebra map $\psi: A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ is dual of a coalgebra map, as we deduce by:

Proposition 3.26. *If $|T_0| \leq 2^{|K|}$ then every algebra map $\psi: A(\mathbb{T}_0) \to A(\mathbb{S}_0)$ is dual of a coalgebra map.*

Proof. It is a trivial consequence of Proposition 3.25 and of the following classical result. □

Proposition 3.27. *Let A, B be sets. If $|B| \leq 2^{|A|}$ then there exists no $|A|^+$-complete non principal ultrafilter in $\mathbb{P}(B)$.*

Combining the previous propositions we obtain a generalization of Leroux's result (see [8]) about the continuous endomorphisms of the Incidence Algebra of a finitely generated Möbius Category.

Proposition 3.28. *Let $\mathbb{T}$ be a finitely generated decomposition structure. Then every automorphism of $A(\mathbb{T})$ is continuous. If $|T_0| \leq 2^{|K|}$ then every endomorphism of $A(\mathbb{T})$ is continuous.*

The first result of this kind was obtained by Baclawski in [1] for Incidence Algebras of locally finite posets. In the same paper it is shown that:
"Every derivation of an Incidence Algebra of a locally finite poset is continuous".

Now, we want to prove a similar result for Incidence Algebras of finitely generated decomposition structures.

We shall denote $\mathbb{D}(\mathbb{S})$ the space of the K-linear maps $D: A(\mathbb{S}) \to A(\mathbb{S})$ satisfying, for all $f, g \in A(\mathbb{S})$,

$$D(f * g) = f * D(g) + D(f) * g.$$

The elements of $\mathbb{D}(\mathbb{S})$ will be called *K-derivations* (or simply derivations) of $A(\mathbb{S})$.

Proposition 3.29. *Let D be a derivation of $A(\mathbb{S})$. Then, for each idempotent element f, $D(f) \in J_1(\mathbb{S})$.*

Proof. Let f be an idempotent element of $A(\mathbb{S})$. If $r \in S_0$ then

$$D(f)(x_r) = D(f * f)(x_r) = 2f(x_r)D(f)(x_r).$$

Therefore, since $f(x_r) \neq \frac{1}{2}$, $D(f)(x_r) = 0$. $\square$

A derivation D of $A(\mathbb{S})$ will be said to be an *inner derivation* with respect to $g \in A(\mathbb{S})$ if, for each $f \in A(\mathbb{S})$, $D(f) = g * f - f * g$. The inner derivation with respect to g will be denoted D_g.

The following proposition about the inner derivations is central. Its proof is inspired by that of a proposition due to Baclawski [1].

Proposition 3.30. *Let $\mathbb{S}$ be a hereditarily finite decomposition structure, and D a derivation of $A(\mathbb{S})$. Then there exists an inner derivation D_g such that $D(x^u) = D_g(x^u)$ for each $u \in S_0$.*

Proof. Notice that $(D(x^s) * x^s)_{s \in S_0}$ is a summable family, thus $g = \sum_{s \in S_0} D(x^s) * x^s$ is an element of $A(\mathbb{S})$. Now, if $u \in S_0$ we have:

$$D_g(x^u) = g * x^u - x^u * g = D(x^u) * x^u - \sum_{s \in S_0} x^u * D(x^s) * x^s.$$

Since

$$x^u * D(x^s) * x^s = \begin{cases} -D(x^u) * x^s & \text{if } u \neq s \\ 0 & \text{if } u = s, \end{cases}$$

$$\begin{aligned} D_g(x^u) &= D(x^u) * x^u + \sum_{s \neq u} D(x^u) * x^s = D(x^u) * x^u + D(x^u) * (\varepsilon - x^u) \\ &= D(x^u). \quad \square \end{aligned}$$

Corollary 3.31. *Let $\mathbb{S}$ be a hereditarily finite decomposition structure and D a derivation of $A(\mathbb{S})$. If $D(x^s)(x_r) \neq 0$ then*

$$\partial_1(s) = \partial_1(r) \quad \textit{and} \quad \partial_0(s) \in [\partial_0(r), \partial_1(r)]$$

or

$$\partial_0(s) = \partial_0(r) \quad \textit{and} \quad \partial_1(s) \in [\partial_0(r), \partial_1(r)].$$

Proof. If we put $D_0 = D - D_g$ and if $D(x^s)(x_r) \neq 0$ we see that:

$$\sum_{a,b} \begin{bmatrix} r \\ a,\ b \end{bmatrix} g(x_a)x^s(x_b) \neq 0 \quad \text{or} \quad \sum_{a,b} \begin{bmatrix} r \\ a,\ b \end{bmatrix} x^s(x_a)g(x_b) \neq 0$$

or

$$D_0(x^s)(x_r) = (x^{\partial_0(s)} * D_0(x^s) * x^{\partial_1(s)})(x_r) \neq 0.$$

Thus the corollary holds. □

Let us now suppose that $\mathbb{S}$ is a finitely generated decomposition structure. Under this hypothesis we prove a proposition similar to Proposition 3.14.

Proposition 3.32. *Let $\mathbb{S}$ be a finitely generated decomposition structure. If D is a derivation of $A(\mathbb{S})$ then, for each pair $u, v \in S_0$ and for each $n \geq 1$*

$$D(x^u * J_n(\mathbb{S}) * x^v) \subseteq (J_1(\mathbb{S}))^{n-1}.$$

Proof. If $f \in x^u * J_n(\mathbb{S}) * x^v$, by Proposition 3.6, f is a finite sum of expressions such as $x^u * f_1 * \cdots * f_n * x^v$, $f_i \in J_1(\mathbb{S})$. Thus, $D(f)$ is a finite sum of expressions like

$$D(x^u) * f_1 * \cdots * f_n * x^v + x^u * D(f_1) * \cdots * f_n * x^v + \cdots$$
$$+ x^u * f_1 * \cdots * f_n * D(x^v)$$

where every addendum is in $(J_1(\mathbb{S}))^{n-1}$. □

Corollary 3.33. *Under the hypotheses of Proposition* 3.32 *if* $D(x^s)(x_r) \neq 0$ *then* $l(s) \leq l(r) + 1$.

Proof. Let r be an element of S such that $D(x^s)(x_r) \neq 0$ and $l(r) = m$. If $l(s) = n$ then $x^s \in x^{\partial_0(s)} * J_n(\mathbb{S}) * x^{\partial_1(s)}$. Therefore $D(x^s) \in (J_1(\mathbb{S}))^{n-1}$. Thus, if $m < l(s) - 1$ then $D(x^s)(x_r) = 0$. □

Finally we can prove our main result about the derivations of $A(\mathbb{S})$.

Proposition 3.34. *Let $\mathbb{S}$ be a finitely generated decomposition structure. If $D: A(\mathbb{S}) \to A(\mathbb{S})$ is a derivation then D is continuous.*

Proof. By Proposition 3.30, we can write $D = D_g + D_0$ where $g = \sum_{s \in S_0} D(x^s) * x^s$. Since D_g, like every inner derivation, is continuous, to conclude the proof it is sufficient to prove that D_0 is a continuous derivation. At first we observe that if $r \in S_m$ and $D_0(x^s)(x_r) \neq 0$ then $(x^{\partial_0(s)} * D_0(x^s) * x^{\partial_1(s)})(x_r) \neq 0$. Therefore by Corollary 3.33, the elements $s \in S$ such that $D_0(x^s)(x_r) \neq 0$ are elements of $S_{m+1}(\partial_0(r), \partial_1(r))$ which is a finite set. Now, let $(k_s)_{s \in S}$ be a family of elements of

K. We have:

$$\begin{aligned}D_0\Big(\sum_{s\in S} k_s x^s\Big)(x_r) &= \Big(x^{\partial_0(r)} * D_0\Big(\sum_{s\in S} k_s x^s\Big) * x^{\partial_1(r)}\Big)(x_r)\\ &= D_0\Big(x^{\partial_0(r)} * \sum_{s\in S} k_s x^s * x^{\partial_1(r)}\Big)(x_r) = D_0\Big(\sum_{s\in S(\partial_0(r),\partial_1(r))} k_s x^s\Big)(x_r)\\ &= D_0\Big(\sum_{s\in S_{m+1}(\partial_0(r),\partial_1(r))} k_s x^s\Big)(x_r) + D_0\Big(\sum_{s\in (S-S_{m+1})(\partial_0(r),\partial_1(r))} k_s x^s\Big)(x_r).\end{aligned}$$

Since

$$\sum_{s\in (S-S_{m+1})(\partial_0(r),\partial_1(r))} k_s x^s \in x^{\partial_0(r)} * J_{m+2}(\mathbb{S}) * x^{\partial_1(r)},$$

by Proposition 3.32,

$$D_0\Big(\sum_{s\in (S-S_{m+1})(\partial_0(r),\partial_1(r))} k_s x^s\Big) \in J_{m+1}(\mathbb{S}).$$

Hence

$$\begin{aligned}D_0\Big(\sum_{s\in S} k_s x^s\Big)(x_r) &= D_0\Big(\sum_{s\in S_{m+1}(\partial_0(r),\partial_1(r))} k_s x^s\Big)(x_r)\\ &= \sum_{s\in S_{m+1}(\partial_0(r),\partial_1(r))} k_s D_0(x^s)(x_r) = \sum_{s\in S} k_s D_0(x^s)(x_r). \quad \Box\end{aligned}$$

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